A Didactic Proposal to Develop Critical Thinking in Mathematics: The Case of Tomás

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Abstract
Critical thinking is an essential skill to be developed in school with every student. As it is a complex and difficult skill to accomplish, it should be introduced early on in a child’s education. A qualitative case study was conducted to analyse the way in which grade 6 students solved mathematical problems to see which aspects of critical thinking were used and developed. Although the study was carried out within a short time frame, it was possible to detect characteristic aspects of critical thinking used by students, which shows that there is a need for more research to be done.

Key words: Basic education, critical thinking, didactic proposal, mathematics problems.

Introduction
Critical thinking, problem solving, creativity and innovation, communication and collaboration has been labelled a fundamental skill (OECD, 2009). Teaching students to think critically should be the responsibility of schools (e.g. Dewey, 1933; van Gelder, 2005; Scriven, 1985; Willingham, 2007), especially schools in the 21st century, as future generations will face challenges that will require them to be able to analyse situations and to make decisions (Marian & Halpern, 2011).

Schools should provide learning environments that help every child at every level of schooling to develop the ability to think critically, since this ability does not develop naturally. It can be integrated into certain subject areas, or taught more generally. In associating critical thinking with the skill of problem solving, an obvious subject choice for analysis is maths, as it is one of the areas of the curriculum that better lends itself to the development of this skill. Interpretation of information, analysis and construction of arguments and drawing conclusions are some aspects of critical thinking (e.g. Ennis, 2011; Lai, 2011) that should be developed in every student (Marian & Halpern, 2011). In Portugal, students aged between 10 and 12 have shown difficulty in problem solving, in the construction of mathematical arguments, in communicating their ideas and findings and in using critical thinking to analyse the plausibility of their answers (GAVE, 2009). To overcome these difficulties, some researchers suggest that their critical thinking should be mobilized (e.g. Ennis, 2011; Fiúza, 2010).

In order to gain knowledge about the development of critical thinking in students of this age group, a study was developed with the aim to analyse how grade 6 students solve mathematical problems and to understand the features of critical thinking that students used in the process of problem solving. To guide the study, two questions were formulated; (a) Which critical thinking skills are revealed by grade 6 students when
solving mathematical problems? (b) Which difficulties are revealed by the students while solving problems?

Definitions of Critical Thinking

There are various definitions of Critical Thinking in the literature, due to the different viewpoints of different areas of study - philosophy, psychology and education (Lai, 2011; Sternberg, 1986).

In the philosophical approach, based on the writings of Socrates and the Socratic method, Critical Thinking is related to the generic characteristics of a person, in what they are capable of doing (Sternberg, 1986), more than their actions or behaviours. A Critical Thinker is described as “someone who is inquisitive in nature, open-minded, flexible, and fair-minded, has a desire to be well-informed, understands diverse viewpoints and is willing to both suspend judgment and to consider other perspectives” (Lai, 2011, p.5).

According to Ennis (2011), “Critical thinking is reasonable and reflective thinking focused on deciding what to believe or do” (2011, p.1). Lipman (1998) states “Critical Thinking is skilful, responsible thinking that facilitates good judgement because it relies upon criteria, is self-correcting and is sensitive to context” (p.38).

For Paul and Elder (2006) “Critical Thinking is the art of analysing and evaluating thinking with a view to improving it” (p.4).

In the psychological approach Brunner and Sternberg are leading names in the field. In this subject area, Critical Thinking focuses on what a person actually does (which depends on the limitations of the person and the environment), indicates types of actions or behaviours Critical Thinkers tend to show (Lewis & Smith, 1993) and connects Critical Thinking with problem solving (Reed, 1998). Willingham (2007) sees Critical Thinking as consisting of “seeing both sides of an issue, to being open to new evidence that disconfirms your ideas, reasoning dispassionately…deducing and inferring conclusions from available facts [and] solving problems” (p.8).

For Sternberg (1986) “Critical Thinking comprises the mental processes, strategies and representations people use to solve problems, make decisions, and learn new concepts” (1986, p.2).

In the educational approach, Critical Thinking combines aspects from both of the previous approaches. Bloom and Dewey are key figures; the latter can even be considered the father of modern Critical Thinking and based his work on observations and experiences in the classroom. Dewey refers to Critical Thinking as “‘reflective thinking’ and defines it as active, persistent, and careful consideration of a belief or supposed form of knowledge” (Fisher, 2009, p.2).

Halpern (1997), defines Critical Thinking as “purposeful, reasoned, and goal directed – the kind of thinking involved in solving problems, formulating inferences, … and making decisions” (p.4). This was the definition used in this study. Although Critical Thinking develops slowly and is hard to teach (Willingham, 2007), it can be approached in schools in different ways; from explicit instruction on the skills, dispositions and abilities of Critical Thinking which is unrelated to specific content, to an integrated approach in a specific subject area (Ennis, 1996; Marian & Halpern, 2011). The process requires the active involvement of the students where it is essential that they think, look for information, question themselves, persist and make an effort. Reason and truth are
valued, along with respecting others in discussion, being willing to look at issues from others’ points of view, acceptance of different perspectives in dealing with situations, distinguishing between definitions and empirical statements, asking for clarification and considering alternatives before making decisions (Bailin, Case, Coombs & Daniels, 1999).

For Halpern (2014) and Marian and Halpern (2011), the development of students’ Critical Thinking skills must take place persistently, in different contexts and be integrated into different areas, although they defend the view that it should also be explored specifically. They have proposed an instructional program in which students should (1) explicitly learn Critical Thinking skills; (2) develop a willingness to work hard at thinking and learning; (3) learn through activities that facilitate the transfer of developed skills; (4) explicitly monitor their actions using metacognitive strategies.

**Convergences and divergences in the different approaches to Critical Thinking**

Although there are many approaches to Critical Thinking, there are overlapping factors (Lai, 2011) when it comes to abilities and dispositions, understood as attitudes or habits of mind, and the importance of background knowledge. Abilities include aspects such as the analysis of arguments, evidence or allegations; making deductive or inductive inferences; judging or evaluating; making decisions or solving problems; asking or answering questions for clarification; defining terms; interpreting and explaining; and seeing both sides of an issue. Dispositions include open and fair mindedness; the propensity to seek reason; inquisitiveness; the desire to be well informed; flexibility; and respect for others’ viewpoints. Background knowledge is essential if students are to demonstrate their critical thinking skills, as to think critically, students need something to think critically about (e.g. Willingham, 2007). “Domain-specific knowledge is indispensable to critical thinking because the kinds of explanations, evaluations, and evidence that are most highly valued vary from one domain to another” (Lai, 2011, p.11).

Ennis (2011) identified six general abilities that a critical thinker should have: (a) basic clarification: focus on a question, analyse arguments, ask clarification questions; (b) decision: judge the credibility of a source, the existence of contradictions, judge observation reports; (c) inference: deduce and judge deduction, make inferences, make and judge value judgments; (d) advanced clarification: define terms and judge definitions; (e) supposition and integration: consider and reason from premises, reasons, assumptions, positions with which they disagree or about which they are in doubt, without letting the disagreement or doubt interfere with their thinking and integrate the dispositions and other abilities in making and defending a decision; (f) auxiliary abilities: proceed in an orderly manner, monitor their own thinking, be aware of unfounded reasoning, and be sensitive to the well-being of others.

According to Ennis (2011) critical thinkers are disposed to: (a) Care about their background knowledge, that it is correct, and that their corrections and decisions are justified. They seek alternative hypotheses, explanations, conclusions, and are open to seeing different sides of an issue; consider seriously points of view other than their own; try to be well informed. (b) Understand and present a position honestly and clearly, and expect the same from others. Be clear about the intended meaning of what is said or
written; listen to others’ arguments, look for conclusions; seek and offer reasons. (c) Care about others. Avoid intimidating or confusing others, taking into account others’ feelings, understanding and welfare.

Different perspectives differ when it comes to the role of dispositions; to the particularities of the subject areas; to the transferability and to the importance of the criteria of Critical Thinking. According to Lai (2011) some researchers agreed that critical thinking is synonymous of “good thinking,” in the sense that truly critical thought can only be exhibited by those with the ability to think critically, who want to and who do so ethically. A person who is capable of thinking critically and chooses not to do so, or does so unethically is not a critical thinker. It is also argued that critical thinking must be developed in the context of a specific domain, as it is possible to think like a scientist or like a historian (e.g. Ennis, 1996; Willingham, 2007) and because of this, the transfer of critical thinking skills between subject areas is seen as being very difficult. “Near” transference is more easily accepted than transference to an entirely new, different discipline. The criteria used to make judgements, support decisions, evaluate arguments, evidence and others’ positions and evaluate one’s own thoughts (Lay, 2011; Lipman, 1988) depend on the domain of interest.

Methodology

Considering the aim of this study and the questions formulated, a qualitative methodology was adopted and a case-study research method was implemented (Coutinho, 2014), during six months.

The participants of the study were a class of grade 6 students. These students were not used to approaching tasks from a Critical Thinking perspective. All of the students in the class solved all of the problems, but the study focused on four cases. To determine the participants for the cases, five tasks were presented to all, three solved in pairs and two individually. The ease with which students communicate, corrected and detailed the answer allowed the selection of the cases – two girls and two boys. The collection of data was done through tasks, observation, written records and video recordings.

As students should be challenged to think critically by doing tasks that require critical thinking, for example tasks that may contain the solutions to other problems or the problem at hand and which the children should think about, (e.g. Ennis, 1996, 2011 Halpern, 1997, 2014; Sternberg, 1986) a didactic proposal was drawn up, organized in three phases: phase 1) A critical analysis of the solutions to problems, solved by someone they do not know; phase 2) Problem solving and a critical analysis of classmates’ solutions to problems; phase 3) Problem solving and a critical analysis of their own solutions to problems.

Phase 1. Two problems that had been solved by students of the grade 6, but from a different class, were given to the students with the aim of developing the students’ decision-making skills. Five different solutions were provided for each problem. The objective was that the students would look at the problem and choose correct resolutions, justifying their choices and explaining how the problem was solved.
Phase 2. The aim was to start analysing the students’ problem solving skills as well as their ability to critically judge resolutions of anonymous colleagues – other students in their class. Three problems were given to the students, who then went on to solve them. The resolutions were then given out to different students, so that they had resolutions to the same problem from colleagues, without knowing whose exactly they had.

Phase 3. In this phase, three tasks were given to the case-study participants for them to resolve and then analyse critically. This phase took place outside the classroom so that students could think aloud, to voice their thought processes and therefore explain their choices and the result that they came to. The aim was to analyse their problem-solving skills, but also their ability to self-criticize.

Analysis criteria were created for each phase, based on ideas from Ennis (2011), Paul and Elder (2006) and data collected. For phase 1 - *A critical analysis of the solutions to problems, solved by someone they do not know* – the following criteria were considered: the ability to evaluate alternatives; the ability to recognise inconsistencies in solutions; analyse and evaluate arguments; reach and evaluate conclusions; identify alternatives and the ability to ensure fairness. For phase 2 - *Problem solving and a critical analysis of classmates’ solutions to problems* – were considered the criteria: the ability to identify and clarify a problem; to form hypotheses; present and evaluate solutions; establish conclusions; distinguish relevant from irrelevant facts; recognise contradictions; ensure fairness; develop intellectual humility and to learn to stop making value judgements. As they had to judge their peers’ answers, who although anonymous were still their colleagues and therefore very close, effective strategies were introduced to enable an analysis of the students’ ability to be impartial. For phase 3 - *Problem solving and a critical analysis of their own solutions to the problems* – were considered the criteria: the ability to identify and clarify the problem, formulate hypotheses, form new ideas, solve the problem, formulate alternative solutions, choose the best solutions, evaluate the chosen answer, develop intellectual humility, stop making value judgements and the ability to develop intellectual courage. The latter is related to the need for pupils to appreciate their own work. It was hoped that they would be able to detect any flaws in their own previous resolution.

**Presentation and Analysis of Results**

Tomás was chosen as a case study to be presented, as well as some of the problems proposed for each of the phases of the didactic proposal.

In phase 1, one of the task given to the students was Task 6.

**Phase 1 - Task 6**

> Francisco and João’s mom bought 5l of milk. Every day Francisco drinks 1/2l and João drinks 3/4l. How many days will it take for the two brothers to drink the 5l of milk? Explain your reasoning. You can use words, diagrams or calculations.

Critically analyse the five answers presented: A, B, C, D and E.
1. Select the correct answers. Justify your choice.
2. Explain the reasoning used in each one of the answers

Each student had to look at 5 different answers (Figure 1).
“A-Answer: They drank in 5 days.”

“C-Answer: 5 days.”

“E-Answer: They together drank the milk in 4 days.”

D-Answer: During 6 days they drank 5 l of milk.

Figure 1. Answers to Task 6
Two out of the five answers given were correct, B and E, but Tomás only identified E as being correct. Tomás’ analysis of answer E in figure 1. “E) This boy added the number of litres drank by the two boys, and subtracted what they drank from 5 litres each day which will give him the correct number of days. This one is correct”.

Tomás did not justify why he thought answer B was wrong. “B) He replaces fractions with decimals. Then he adds the results together until he reaches 5l. I think he’s wrong”. He analysed the other answers, signalling the mistakes, but did not indicate whether he thought they were correct or incorrect. Tomás pointed out some mistakes he identify in the answers for Task 6.

But $5 \times 5$ isn’t 20. It’s 25

But I don’t know how he/she knows that João drinks half of the milk and Francisco the other half

D) This boy divided the milk, $\frac{1}{2}$: half of the milk and $\frac{3}{4}$ : the other half divided into 4 with 3 of the quarters coloured in. I don’t know how he knows that a half is one day, as both of the boys drink milk every day, not just Francisco ”.

To explain the thinking of each of the students solving the problem, Tomás signalled what he thought was the right method (Figure 5).

After that Tomás presented explanations to the analysed answers.

A) This boy though to add how [many milk] they drink together, after subtract from 5 liters. Remove from 5 liters the amount drank by each one, but pay attention $5 \times 5$ is not 20.

B) This [boy] though to replace $\frac{3}{4}$ and $\frac{1}{2}$ by each result (0.75 and 0.5). After add the liters of each boy and calculate till get 5 liters.

C) This [boy] thought to design 5 milk box, and divide the milk box in two, one to João and the other to Francisco.

D) This boy though in dividing the milk box, $\frac{1}{2}$= half of the box and $\frac{3}{4}$= from the other half, divide in four and paint 3.

E) This boy though to add the liters that the two boys drank together, after subtract from 5 L the liters drank by the two boys and he got the right number of days!

Tomás identified the correct and incorrect answers, but made a judgment error. However, he analysed the answers given to him, evaluated correct and incorrect alternatives, justified his choice, analysed or evaluated arguments, recognised inconsistencies in the answers, and identified alternatives. In this phase he did not demonstrate a profound level of critical thinking.

In phase 2, one of the given tasks to the students was Problem 2 – Canoeing (Figure 2). As was asked of him, he answered the question and then analysed the answer that had been given to him. Tomás interpreted “26 students in Elisa’s class” as not including
Elisa. He considered that the total number of students in the class was 27 and his answer uses this number.

The 26 students in Elisa’s class went canoeing. They needed to rent some canoes. They all went canoeing together, at the same time, and there were no empty spaces.

<table>
<thead>
<tr>
<th>Types of canoe</th>
<th>Number of canoes available</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 person</td>
<td>6</td>
</tr>
<tr>
<td>3 person</td>
<td>5</td>
</tr>
<tr>
<td>4 person</td>
<td>2</td>
</tr>
</tbody>
</table>

Using the information in the table, calculate how many canoes they rented. How many canoes of each size were rented?

*Figure 2. Phase 2 – Problem 2 – Canoeing*

Tomás adequately analysed the situation, drew up a plan and came to a conclusion, as can be seen in Figure 3. As he did not specify the size of the rented canoes, it can be concluded that he either did not understand the question or forgot to do so.

*Figure 3. Tomás’ answer*

Tomás showed that he could identify the general question, clarify some aspects of it, form ideas related to the part that he detailed, choose the most appropriate answer, evaluate the answer chosen and establish some conclusions.

In the second part of this exercise he analysed an answer that was given to him, Figure 4. “Answer: They rent 6 2-person canoes, 2 3-person canoes and 2 4-person canoes”

*Figure 4. Answer to the problem analysed by Tomás.*

As he thought that Elisa’s class had 27 students, Tomás’ conclusion was reached using this number, having judged and evaluated the answer given to him. “The calculations are correct but the problem is wrong. Elisa’s class has 26 students, but Elisa is a student too. So the number of students is 27, and my colleague just used 26 students”.

In this phase, Tomás identified and clarified the problem, chose the best answer, established conclusions, evaluated solutions, analysed and evaluated arguments, and showed that he was being careful in the way in which he pointed out what he thought
was a mistake made by his classmate, showing intellectual humility. There was no significant development of his critical thinking skills compared to the previous phase.

In phase 3 one of the proposed exercises was the Problem: Box of chocolates (Figure 5). Tomás had to solve the problem, think aloud and then evaluate his answer.

For his answer to question 1, Tomás said that he had to use “cross-multiplication: one box is to one euro seventy eight as all of the boxes is to x” and he had to count the number of boxes in the pile. He explained this by saying:

We go one, two, three, four, five, one two, three, four, five. We only count the side ones and then we work out 5x5. It’s 25. Then we count the ones above that, one, two, three, one two, three. 3x3 is 9. Twenty-five plus nine is… it’s thirty-four, and plus one is thirty-five. Now we can use the rule [of cross-multiplication].

He cross-multiplied, used a calculator and got the total cost of the boxes of chocolates: 62.30 €.

"R: The boxes in the pyramid cost 62.3 €"

Next, Tomás went on to question 2. He started by reading it and then explained his thought process in great detail.

Fourth level! … well, this question consists of… well, as the pile only has three levels, the first one on top, the second and the third. In each level… I noticed that the first one only has one; the second has one underneath and then one next to it, then another one next to that, one behind it and another one next to it. One on each side. Then on the other one next to it, there is another one next to it, another one next to it, and another one next to it. So, the next one is going to have, to have one box next to each of those. So we have… to… know how many [boxes] we have here. One, two, three, four, five.

He was explaining how he could find out the number of boxes on each level of the pile down to the third level, and then carried on to the fourth level.
Well this one has five, if there has to be one more on this side and one for the length, well five plus two is seven. It will have to be seven boxes long. If it is seven boxes long, this side will also have to have them, we can count one, two, three, four, five on this side, five plus two is seven. So it will be seven long on this side and seven long on this side. So if it is seven boxes long on that side and this side, we have to work out seven times seven to know how many boxes there will be in the next level. I’m not very good with multiplication! Forty-nine. That’s the fourth level, there are forty-nine boxes.

He repeated the thought process and went on to work out the fifth level.

Now we have to find out the fifth. As we added two to these here, this side and that side, and I got seven, we have to add another two onto the side. Well, seven plus two is nine… eight, nine. Well if we have nine on this side, as they are the same length there will be another nine on the other side. So nine times nine, it’s the same… eighty one boxes on the fifth level.

Then he got to the last question. He read it and started thinking aloud straight away.

Tenth level! As we add two to every level, well if we add them together, as it’s this side and that side, all we have to do is add them. We already have level number five, all we do is nine plus two and keep adding two until we get to the tenth level. For level number five we added two to nine and the result was the number in the sixth level. Next we add two to the answer and get the seventh level, and just carry on doing that.

Figure 7 shows what he wrote, expressing his thoughts and answering the question.

![Figure 7. Answer to question 3 from the problem about the Box of chocolates.](image)

Figure 7. Answer to question 3 from the problem about the Box of chocolates.

Tomás identified and clarified the problem, formed new detailed ideas and fundamentally, explained these ideas orally and in writing, solved the problem, established conclusions and showed intellectual courage when he openly said that he isn’t “very good with multiplication”.

**Conclusions**

Having concluded the phases of the didactic proposal, in which grade 6 students developed critical thinking skills in Maths, it can be said that Tomás’ ability to analyse critically improved as he went through the phases analysing and solving problems. He stopped using superficial descriptions and started to use more detailed ones, where he explained what he was thinking in detail.

According to Ennis (2011) he showed critical thinking skills such as: evaluating alternatives; recognising inconsistencies; analysing and evaluating arguments; a sense of equality; identifying and clarifying problems; formulating hypotheses; putting forward solutions and evaluating them; establishing conclusions; solving and analysing his own answers critically; choosing the best answer; evaluating his own answer; he
developed a sense of intellectual humility, stopped using value judgements and developed intellectual courage. As time went on, he revealed more developed skills, showing that he had reached the inference phase (Ennis, 2011), as well as showing emotional sensitivity towards his colleagues.

He had difficulty in analysing the adequacy of the answers given by his peers, in judging and evaluating the adequacy of his own answers and solutions, in thinking about his own thought process, explaining his reasoning, using mathematical terminology and even communicating his ideas to clarify his reasoning. These difficulties seem to arise due to the lack of practice in doing problems that require critical thinking, since this skill is neither easy nor quick to develop (Willingham, 2007). Therefore it is important to give students continual opportunities to allow for this development; challenges to increase their knowledge of maths, solve problems, share ideas, explain their reasoning, actively listen to their colleagues in the class and think about their own thought processes. The use of open questions and the implementation of projects can be used for the development of critical thinking, as they supply students with the need to make decisions through the pressure of the discussion of ideas and in the integration of opinions that are well backed-up, even if they are different to those held initially. In order to boost the development of Critical Thinking, an improvement is suggested for the didactic proposal used in this study: the implementation of a fourth phase that encompasses the solving of open problems and/or the implementation of projects. In this phase students should work in small groups to share their ideas, to discuss and to get a final resolution of the problem or to concretize the project.

References


