Mathematical reasoning and proof schemes in the early years

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Abstract
Mathematics and reasoning are strongly related. Every child must have the opportunity to reason mathematically, to deepen its mathematical comprehension. To do so children need a daily mathematics diet linked to mathematical reasoning. To understand what kind of reasoning children use, how they justify their options, what difficulties they reveal, a qualitative case study was designed with 2nd grade students. The main results point out that young child reveal emerging deductive reasoning, empirical and analytic proof schemes to support their own resolution options and some difficulties in organizing their work, but they reveal to be persistent to look for solutions.

Key words: Early years, justification, mathematics, proof schemes, reasoning.

Introduction
Since a long time ago, researchers (e.g. Darmstadter, 2013; English, 2004; Goldenberg, Cuoco & Mark, 1998; Harel & Sowder, 1998, 2007; Russel, 1999; Sternberg, 1999; Sumpter & Hedefalk, 2015) and international organizations (see NCTM, 2000, 2014) highlights the importance of mathematical reasoning and the necessity to explore it in all classroom, with all students and related with all mathematical content. The National Council of Teachers of Mathematics (NCTM, 2000) refers that mathematical processes must be develop in the classrooms, since early grades, with all students. Students must explore problem solving and problem posing, explain their reasoning and proof, develop communication, and teachers must highlight different connections inside mathematics and relating mathematics to other subjects and encourage or welcome different representations for the same situation.

In order to gain knowledge about the student’s ways of reasoning, a study was developed with the aim to know how students explain their mathematical reasoning and justify their ideas or results. To guide the study, three questions were formulated (1) How characterize the mathematical reasoning of 2nd grade students when they solve process problems?; (2) How 2nd grade students justify their results?; (3) Which difficulties do the students reveal while they solve process problems?

Mathematical reasoning
As reasoning allows getting conclusions, children must have the opportunities to face tasks that challenged them to reason since early years. Being the foundation of deep understanding in mathematics, since kindergarten, reasoning should be a consistent part of students’ daily mathematical experience (Bragg, Loong, Widjaja, Vale & Herbert, 2015; NCTM, 2000).
Educational programs should enable all students “to recognise reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; select and use various types of reasoning and methods of proof” (NCTM, 2000, p.56).

One of the mathematics teaching practices that NCTM (2014) points out to be used in classroom is to implement tasks and pose purposeful questions that promote reasoning and problem solving. “Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies” (p.10).

Reasoning mathematically is a habit of mind (Goldenberg, Cuoco & Mark, 1998) and like all habits, it must be developed through consistent use in many contexts. This requires that teachers organize lessons “to prompt student interactions and discourse, with the goal of helping students to make sense of mathematical concepts and procedures” (NCTM, 2014, p.10). This has to be done because

Students are reasoning mathematically when they explain their thinking, deduce and justify strategies and conclusions reached, adapt the known to the unknown, transfer learning from one context to another, prove that something is true or false, compare and contrast related ideas and explain their choices (Bragg, Loong, Widjaja, Vale & Herbert, 2015, p.10).

Aligned with several researchers and the position of international associations (e.g. Darmstadter, 2013; Sumpter & Hedefalk, 2015; NCTM, 2014), in Portugal, the previous Portuguese national curriculum (ME, 2007) points out that the main goals of mathematics teaching is to help students to explain their reasoning and to justify. Students should be able to describe orally and by writing their mathematical comprehension and the procedures they use and they should learn to justify their assertions since early grades, using particular examples. All along the time, their justifications should be more general and students must discriminate particular examples from general mathematical arguments. Students must justify their reasoning and the conclusions they reach (ME, 2007). However, nowadays our national curriculum (MEC, 2013) puts the emphasis on deductive reasoning.

Some educators and parents believe that students should be educated in the same way they were, by memorizing facts, formulas and procedures, practicing skills repeatedly, leaving problem solving and mathematical reasoning just for the final years of schooling. These kind of unproductive beliefs are an obstacle to afford students with an effective teaching and learning mathematics, as defended by NCTM (2014). Nevertheless, we also have educators who believe that mathematics lessons should engage students in solving and posing problems, discussing strategies and results, sharing their reasoning, considering others reasoning, explaining their options, justifying and making sense of mathematics content. In a word, doing mathematics (NCTM, 2014). This agree with the new document (ME, 2017), that outlines the student’s competency profile upon leaving compulsory schooling, which highlights reasoning and problem solving.

The way to promote the development of student’s mathematical reasoning is through challenging tasks, that “encourage high-level student thinking and reasoning (…)

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multiple entry points, including the use of different representations and tools, and (...) fostering the use of varied solutions strategies” (NCTM, 2014, p.17).

In this text, to reason is to draw logical conclusions based on evidence, specific assumptions or rules (NCTM, 2000) and justify is to sustain the resolutions based on resistant arguments (Fonseca, 2004).

**Types of mathematical reasoning**

We can address different types of reasoning in mathematics classroom from early years. To explore situations and experiments, looking for patterns brings students closely to inductive reasoning. This will allow them to formulate conjectures, test, and search for counterexamples, refine the conjectures and present justifications, to convince colleagues or teachers of the validity of their findings. This approach puts students closer to deductive reasoning or in the beginning of schooling in the emergent deductive reasoning (Baroody, 1993). Reasoning by analogy assumes vital importance, in the beginning of schooling, because students can develop the ability to reason by making decisions about new things based on similarities and establishing parallels with previous situations (e.g. English, 2004, Goswami, 1992, Sternberg, 1999). Already Pólya (1945), in detailing his method to solve problems, focused on the importance of think about a similar previous solved problem. The use of analogy can make easier the generalizing mode for young students.

According to Herbert, Vale, Bragg, Loong and Wanty (2015), inductive reasoning is a form of reasoning that uses analogies, examples, observations and experiences to form conclusive propositions. It is related to the

Process of reasoning in which a general conclusion is drawn from a set of particular premises, often drawn from experience or experimental evidence (...) is used to make connections among studies, note patterns, fill the gaps, and attempt to explain why contradictory findings should be ignored or downplayed (p.28).

**Deductive reasoning** is the “process of reasoning typical of mathematics and logic, in which a conclusion follows necessarily from given premises so that it cannot be false when the premises are true” (Herbert, Vale, Bragg, Loong & Wanty, 2015, p.28). Primary children use this type of reasoning when they proceed systematically to justify a conjecture. Abductive reasoning is the “process of inferring certain facts/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation” (p.28).

Kilpatrick, Swafford and Findell (2001) refer the adaptive reasoning as one of the strands of mathematical proficiency, along with conceptual understanding, procedural fluency, strategic competence and productive disposition. “This is the capacity for logical though, reflection, explanation, and justification” (p.5) and “define the mathematical reasoning of school-age students” (Herbert, et al., 2015, p.29). The authors refer that adaptive reasoning encompasses the three types of reasoning, inductive, deductive and abductive, and it is present in student’s justifications, explanations and reflections about their own practices.

Children use adaptive reasoning when they bring together facts, procedures, concepts, and solution methods to make sense of a mathematical problem. They begin to reason in this
way when they find examples that satisfy generalizations and disprove conjectures through counter examples (Herbert, et al., 2015, p. 29).

**Justification and proof schemes**

Justification is central in mathematics classroom, can permeate the whole mathematics curriculum, and it stands out whenever students make a verification, communicate to their colleagues and teacher their reasons to sustain their own options (e.g. Harel & Sowder, 1998, 2007; Schultz-Ferrel, Hammond & Robles, 2007).

Justifying or proving a result involves two steps: ascertain and persuading. The first one is to convince oneself and the second to convince others (Harel & Sowder, 1998, 2007). Convincing oneself, those not mean the use of logical or deductive processes, is an easier process, as referred a long time ago by Mason, Burton and Stacey (1982). To persuade others is a most difficult issue, more demanding and in which may be used different arguments from that used to convince oneself.

Students might justify their mathematical results in different ways: *I just know it*; *Teacher told me last year*; *I got some examples*; *We found a pattern*; *You can see it from drawing*; *I think I have a proof* (Harel & Sowder, 1998, 2007). The authors define a proof scheme as “the arguments that a person uses to convince herself or others of the truth or falseness of a mathematical statement” (Harel & Sowder, 1998, p.140). The arguments used by students can be organize into three categories: (a) *External conviction proof schemes*. Students present a justification, convinced themselves and trying to convince others, based in some outside source like an authority (teacher; colleague, scholar book), the appearance of the argument or symbol manipulations; (b) *Empirical proof schemes*. Student’s justifications are make on the basis of one or more examples, direct measurements, substitutions of specific numbers in algebraic expressions or perceptions; (c) *Analytic proof schemes*. Student’s justifications are concerned with the general aspects of a situation.

To Stylianou, Chae and Blanton (2006) the student’s use of organized strategies to solve problems, like an organized list or procedure, concur to the emergence of analytic proof schemes. According to Plaxco (2011)

> at a given time, a person may not completely display evidence of exactly one proof scheme (...) a student could exhibit both empirical and analytical proof schemes within a short period of time (...) such student could be said to have an “emerging analytical” proof scheme (p.7).

The arguments presented by the same person in the same question, or in different contexts or at different times might fall into different categories, but a unique line of reasoning might involve a combination of different proof schemes.

**Methodology**

According to the problem under analysis, the research followed a qualitative case study approach. The study was carry out by Vânia, a young teacher in the first year of her teaching career, in which she was teaching a 2nd grade class (7/8 years old) with 24
children. To analyze possible differences Vânia selected 3 cases, but this paper just refers to “Carlota” case. Data was collected through process problem, participant observations, intentional and informal conversations, audio and video recordings, documents, and field notes. To solve process problems students also need to use strategies, like trial and error, organized list or find a pattern. During the resolution of the tasks, Vânia made observations focusing in the procedures, strategies and difficulties revealed by students. After each task, Vânia had informal conversations with case students in order to clarify reasoning issues, not explicit in the written resolution. Only after this, Vânia directs discussion with the large group, were different resolutions could be presented and discussed.

The didactical intervention follows four steps: (1) problem presentation. Students read the problem and Vânia posed interpretation questions; (2) students individually solve the problem, without any help. During this time Vânia made observations, took notes related to the strategies used, revealed difficulties and presented justifications; (3) intentional conversation. Before global discussion in the class, Vânia had informal conversations with each of the three cases to clarify some issues of the resolution and to deeper understand about their reasoning; (4) global discussion. The last step is the sharing resolution and discussion with whole group.

**Presentation and analysis of results**

These results are relative to a case study, Carlota, a 7 years old child. We choose Carlota because she is autonomous in the individual work and reveal a good level of oral communication. Due to page limitation, we will analyse just three of the process problems solved in the study.

**Task 2**

*Pimpão is an assistant of Santa Claus. He had just one green wool cap and a white scarf. As he is very responsible and hardworking, Santa Claus decided to give him a gift: two wool caps (blue and orange) and three scarves (purple, dark red and light green). Now he has three wool caps - dark green, blue and orange, and four scarves - dark red, purple, white and light green. How many outfit, including one cap and one scarf, can Pimpão organize?*

Carlota read the problem and understood the request. She manipulated the available concrete material (caps and scarves), combined, in an organized way, each caps with the four scarves and she used an organized list to solve the problem (Figure 1).

*Figure 1. Carlota’ answer to Task 2: 12 different outfits (Esteves, 2013, p.63).*
During the conversation, Carlota explained how to obtain the number of outfits without drawing, revealing critical thinking because she considered the conditions of the problem and crossed them correctly to get the solution. In addition to the organized list, she explained how she could use repeated addition “4+4+4”, which she easily transformed in a multiplication, revealing to understand the additive concept of multiplication, revealing emergent deductive reasoning (Baroody, 1993). She used examples, but also consistent explanations. Her work revealed the emerging analytic proof scheme (Harel & Sowder, 2007; Plaxco, 2011). Carlota was sure of her answer.

**Task 3**

*Teachers organized a Christmas party for her 24 students. They gave to each student a goodies bag with 3 chocolate tablet squares, 1 Santa Claus chocolate, 2 sweets. They buy 8 Chocolate tablets, 3 boxes with Santa Claus shaped chocolate and 2 sweets bags.*

![Chocolate tablet](image1) ![Santa Claus chocolate](image2) ![Sweets bag (20)](image3)

*How many goodies bags they could prepare? Is anything left? How many? How many goodies do they need to buy to offer a goodies bag to all students?*

Carlota read the problem and understood the request. She uses drawings to represent data. Her exploration began by the chocolate tablets. She concludes that 8 chocolate tablets could be split by 16 goodies bags (Figure 2).

![Figure 2. Carlota’ answer to Task 3: 16 goodies bags (Esteves, 2013, p.64)](image4)

She follows to the Santa Claus chocolate and concluded that could be organized 18 goodies bags, because she had 18 shaped chocolate (Figure 3) and with the all the 40 sweets, 20 goodies bags could be organized, because in each one she puts 2 sweets.

![Figure 3. Carlota’ answer to Task 3: 18 goodies bag (Esteves, 2013, p.65)](image5)
Her representations reveal a clear and adequate reasoning. Comparing the different results, she conclude the possibility to organize 16 goodies bags, indicating the remaining.

“I just have chocolates for 16 goodies bag. I have left 8 sweets and 2 Santa Claus chocolates”. After this, Carlota explained how many chocolate tablet, shaped chocolate and sweets, are needed to prepare more 8 goodies bags necessary to all students.

We need 8 bags [more] to have goodies bags for all [24 students]. It is necessary to buy more 4 Chocolate tablets, 1 Chocolate package (6 Santa Claus), because 2 are left, 1 sweets bag, but it will be necessary only 8 sweets because 8 [sweets] are left.

Carlota used drawings and logic deduction strategies. Beyond drawings and logic deduction, she presented an oral explanation. She explored, understood the results obtained, explained her options and expressed the emerging analytical proof scheme (Plaxco, 2011). She followed the conditions and used reasoning and mental operations to support her conclusions. Difficulties not revealed. She expressed emerging deductive reasoning to solve the task (Baroody, 1993), because she pointed, in detail, the number of goodies per bag. Once again, her narrative seems to follow the pattern "if ... then ..." of deductive reasoning

[if] ... are 8 chocolate tablet as it says here and ... I divided into 3, and [then] 3 plus 3, 3 plus 3, 3 plus 3, 3 plus 3, 3 plus 3, 3 plus 3, 3 plus 3, 3 plus 3, [then] is equal to 16 [goodies bags that can be prepared].

Carlota was sure of her answer.

**Task 5**

*Ulises has a chicken. A certain day he found one egg in the hennery. He brought the egg to his house and packed it in an empty egg box (6 eggs). In how many different ways Ulises can put the egg in the box?*

*Next day he found two eggs in the hennery. In how many different ways Ulises could pack the two eggs in an equal empty box? If he picks three eggs, in how many different ways could them be packed?*

Carlota read the problem and understood the request. She manipulated the available concrete material. She used a drawing to represent all the possibilities to solve the problem (Figure 4).

![Figure 4. Carlota’ answer to Task 5: one egg in the box (Esteves, 2013, p. 71).](image)

First, I counted the places. I counted six. As I just have one egg, I put it in each one of the empty spaces: in the first, in the second, in the third, in the fourth, in the fifth and in the sixth. Six different possibilities.
Carlota was sure of her answer.
She began the second part of the task with the manipulatives (Figure 5).

![Figure 5](image)

*Figure 5. Carlota manipulated the concrete material in Task 5 (Esteves, 2013, p. 72).*

Carlota made experiences, use trial and error strategy. She presented in the grid (Figure 6) all the different possibilities to display two eggs in the box, but she did this by chance!

![Figure 6](image)

*Figure 6. Carlota’ answer to Task 5: two eggs in the box (Esteves, 2013, p. 72).*

She revealed difficulties with organization. For this reason, she is not certain that have presented all the possibilities. During the conversation, Carlota explained how she though

in the 1st and in the 2nd place, after in the 2nd and in the 3rd, after in the 1st and in the 4th, the two 1st [in the 1st top line and in the 1st bottom line], after in diagonal and in … the other [his symmetric], finally in the middle, one in the top line and other in the bottom line. I think there is nothing missing.

Carlota began to solve the third part of the problem, using concrete manipulatives displayed.

![Figure 7](image)

*Figure 7. Carlota’ answer to Task 5: three eggs in the box (Esteves, 2013, p. 73).*

Carlota made experiences. She used trial and error strategy. The difficulties continued. Carlota revealed difficulties to think in an organized way; she repeated some dispositions (Figure 7, green and blue marks). This problem seems to be very difficult
to Carlota, because she often changed all the eggs from one box to the next. She did not reveal to have understood that it is possible to fix two eggs and move just the third one. Looking to her representations it seems that Carlota had troubles in organizing their own reasoning. Does not organizing representations be a consequence of not organizing reasoning?

Carlota revealed a not organized work and did not explain their options. She thought that she had experiment all the possibilities, but has no confidence in her answer. Trial and error is the strategy used; inductive reasoning and she displayed empirical proof schemes.

In this task Carlota exhibit two different proof schemes, what are consistent with Harel and Sowder (1998).

Conclusions

Carlota read correctly the problems, understood the tasks proposed and completely solved Task 2, Task 3, and the first part of Task 5. He used several strategies, like organized lists, drawing, and trial and error. Her work revealed emerging deductive reasoning and emerging and empirical analytic proof schemes. She revealed difficulties in organizing the exploration of the Task 5, what restrain the indication of all the possibilities of egg storage.

The organization she revealed in some tasks gave rise to the emergence of deductive reasoning, as well as the analytical proof scheme (Stylianou, Chae & Blanton, 2006). Nevertheless, there are some open questions: Why did not she organize the experiments in the 2nd and 3rd part of Task 5? Did Carlota need any suggestions? More time? Would the last part of the task be complex, by the large number of variants to consider? Should the organization in the 2nd part contribute to the success in the 3rd?

The results confirm the relationship between organized procedure and the emergence of analytic proof scheme (Stylianou, Chae & Blanton, 2006). It is evident that, even in early years, the learning environment must arouse students to an organized work, contributing to mental organization in order to give students access to more consistent reasoning and means of justification, to help them to develop a deeper understanding of mathematics.

These results shows that from early years students reveal analytic proof schemes (Harel & Sowder, 2007), which means that they could focus in the general issues of a mathematical situation and they can begin to construct general mathematical arguments. To develop mathematical reasoning of all students, in all grades and in all contents it is necessary to create challenging learning environments where students has the opportunity to work daily with tasks that contribute to experiment, observe, look for patterns, make conjectures, explain the conjectures, discuss, argument and justify their reasoning.

Developing students' reasoning from early years is possible and is the school's mission.
References


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